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Some Remarks on the Smoluchowski–Kramers Approximation

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According to the Smoluchowski–Kramers approximation, solution q_t^μ of the equation $\mu \ddot{q}_t^\mu = b(q_t^\mu) - \dot{q}_t^\mu + \sigma(q_t^\mu)\dot{W}_t, q_0 = q, \dot{q}_0 = p$, where $\dot{W}_t$ is the White noise, converges to the solution of equation $\dot{q}_t = b(q_t) + \sigma(q_t)\dot{W}_t, q_0 = q$ as $\mu \downarrow 0$. Many asymptotic problems for the last equation were studied in recent years. We consider relations between asymptotics for the first order equation and the original second order equation. Homogenization, large deviations and stochastic resonance, approximation of Brownian motion W_t by a smooth stochastic process, stationary distributions are considered.

KEY WORDS: Smoluchowski–Kramers approximation; homogenization; large deviations, stochastic resonance

1. INTRODUCTION

The motion of a particle of mass μ in a force field $b(q) + \sigma(q)\dot{W}_t$ with the friction proportional to the velocity is defined by the Newton law:

$$\mu \ddot{q}_t^\mu = b(q_t^\mu) + \sigma(q_t^\mu)\dot{W}_t - \alpha \dot{q}_t^\mu, \qquad q_0^\mu = q \in R^n, \qquad \dot{q}_0^\mu = p \in R^n. \tag{1}$$

Here $b(q)$ is the deterministic part of the force, $\dot{W}_t$ is the standard Gaussian white noise in R^n, $\sigma(q)$ is an $n \times n$-matrix. The coefficients $b(q)$ and $\sigma(q)$ are assumed to be regular enough, so that the solution of (1) exists and is unique. The term $\alpha \dot{q}^\mu$ describes the resistance(friction) to the motion. First, we assume that the friction coefficient α is a fixed positive constant. Then, without loss of generality, one can put $\alpha = 1$. Rewriting (1) as a system, we have (for $\alpha = 1$)

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$$\dot{p}_t^\mu = \frac{1}{\mu} b(q_t^\mu) - \frac{1}{\mu} p_t^\mu + \frac{1}{\mu} \sigma(q_t^\mu) \dot{W}_t, \quad p_0^\mu = p, \quad \dot{q}_t^\mu = p_t^\mu, \quad q_0^\mu = q. \tag{2}$$

Together with system (2), consider the stochastic differential equation

$$\dot{q}_t = b(q_t) + \sigma(q_t) \dot{W}_t, \qquad q_0 = q \in R^n. \tag{3}$$

One can prove, that, for any $T > 0$ and $p, q \in R^n$,

$$\lim_{\mu \downarrow 0} \max_{0 \leqslant t \leqslant T} |q_t^\mu - q_t| = 0, \tag{4}$$

say, in probability. This statement is called Smoluclowski–Kramers approximation (of q_t^μ by q_t).[1–6] This result is the main justification for using first order equation (3) to describe the small particle motion. But essential part of modern research related to equation (3) concerns asymptotic problems. For example, behavior of the stochastic process defined by (3) as $t \to \infty$, and its stationary distribution are of interest. Another example is given by the homogenization problem for Eq. (3): Suppose, that $b(x)$ and $\sigma(x)$ are periodic with a small period ε. As is known, one can introduce in this case a diffusion process with constant diffusion and drift coefficients which approximate the process with periodic coefficients. A similar result holds if $b(q)$ and $\sigma(q)$ in (3) are space-homogenuous random fields. But how is this approximation related to the process defined by Eq. (2)? How is the stationary distribution for (3) related to stationary distribution of the process q_t^μ defined in (2)? Statement (4) says nothing on these relations.

Various large deviation problems for Eq. (3) were considered in recent years. For example, if the matrix $\sigma(q)$ is replaced by $\sqrt{\varepsilon}$ multiplied by the unit matrix, then exit problems and stochastic resonance[7] for the process $q_t = q_t^\varepsilon$ are of interest. How are these results for q_t^ε and for $q_t^{\mu,\varepsilon}$ related. Another large deviation problem concerns the occupation times.

It is well known,[8] that if the Wiener process W_t in (3) is replaced by a smooth process V_t^δ approximating W_t as $\delta \downarrow 0$, then the solution q_t^δ of the modified equation (3) converges as $\delta \downarrow 0$ to the solution of Stratonovich's equation

$$\dot{\tilde{q}}_t = b(\tilde{q}_t) + \sigma(\tilde{q}_t) \circ \dot{W}_t, \qquad \tilde{q}_0 = q. \tag{5}$$

But it is natural to put V_t^δ instead of W_t in (1) or (2), and to consider the limit of $q_t^{\mu.\delta}$ as both, μ and δ, tend to zero.

We will see that the answer to some of these questions is not trivial: The limit can depend on the way how the parameters approach zero. On the other hand, we will see that, in certain problems, the Smoluchowski–Kramers approximation is better than it is guaranteed by equality (4).

One should mention that the deterministic component of the force in (1) can depend not just on q_t but on $\dot{q}_t$ as well (then the term $-\alpha\dot{q}_t$ in (1) can be omitted):

$$\mu\ddot{q}_t^\mu = b(\dot{q}_t^\mu, q_t^\mu) + \sigma(q_t^\mu)\dot{W}_t, \quad q_0^\mu = q, \quad \dot{q}_0^\mu = p. \tag{6}$$

If $b(p,q)$ is nonlinear in p, the limit of q_t^μ as $\mu \downarrow 0$, in general, does not exist. But if $\sigma(q) = \varepsilon\tilde{\sigma}(q)$, one can consider the double limit as $\varepsilon, \mu \downarrow 0$. Under certain conditions, such a limit exists and can be used for description of some interesting effects (see[9,10]).

Finally, I mention the case when μ is fixed and the friction and diffusion coefficients in (1) are small. This is an example of perturbations of a Hamiltonian system (an oscillator). Under some assumptions, the motion of the particle in this case has a fast and a slow components, and the limiting slow motion is a diffusion process on a graph or on an open book defined by the first integrals of the system.[11,12]

2. ITO'S INTEGRAL VERSUS STRATONOVICH'S INTEGRAL

The δ-correlated white noise in Eqs (1) or (2) describing motion of a physical particle is an approximation for a more regular (and more realistic) random noise which has a short but not zero correction. As is known, if the Wiener process W_t in (3) is replaced by a process V_t^δ with smooth trajectories which converges to W_t uniformly on a time-interval $[0,T]$ as $\delta \downarrow 0$, and q_t^δ is the solution of such a modified equation, then q_t^δ converges uniformly on $[0,T]$ to the solution of Eq (3) with stochastic term understood on the Stratonovich sense.[5,8] But if q_t^δ is considered as the position of a physical particle at time t, we, actually, should put V_t^δ instead of W_t in Eqs (1) or (2), but not in (3), and consider the two-parameter asymptotic problem as $\mu, \delta \downarrow 0$. The function q_t^δ in (2) is continuously differentiable in t. Therefore, Stratonovich's and Ito's interpretations of Eq. (2) coincide. But which stochastic integral should be considered in (3) as an approximation for q_t^δ, $0 < \mu, \delta \ll 1$? It turns out that the answer depends on the way how μ and δ approach zero.

First, we need some auxiliary bounds. Consider the equation

$$\mu\ddot{q}_t^\mu = b(t, q_t^\mu) + \sigma(t, q_t^\mu)\dot{W}_t - \dot{q}_t^\mu, \qquad q_0^\mu = q, \qquad \dot{q}_0^\mu = p. \tag{7}$$

We assume that $b(t,q)$ and $\sigma(t,q)$ a continuous and Lipschitz continuous in q. Besides, assume, for brevity, that $b(t,q)$ and $\sigma(t,q)$ are bounded uniformly in t and q. Let q_t be defined by equation

$$\dot{q}_t = b(t,q_t) + \sigma(t,q_t)\dot{W}_t, \qquad q_0 = q. \tag{8}$$

Lemma 1. Let q_t^μ and q_t be defined by Eqs (7) and (8), respectively.

(i) Assume, first, that $\sigma(t,q) \equiv 0$, Then for any $T > 0$,

$$\max_{0\leqslant t\leqslant T} |q_t^\mu - q_t| \leqslant e^{KT}\mu[T \sup_{0\leqslant t\leqslant T, q\in R^n} |b(t,q)| + |p|],$$

where K is the Lipschitz constant of $b(t,q)$ in $q \in R^n$.

(ii) There exists $c_1 > 0$ defined by T, n and the Lipschitz constant K of the coefficients $b(t,q)$ and $\sigma(t,q)$, such that

$$\max_{0\leqslant t\leqslant T} |q_t^\mu - q_t|^2 \leqslant c_1\mu[|p|^2 + \| b \|^2 + \| \sigma\sigma^* \|],$$

here $\| b \| = \sup_{0\leqslant t<\infty, q\in R^n, 1\leqslant i\leqslant n} |b^i(t,q)|$, $\| \sigma\sigma^* \| = \sup_{1\leqslant i,j\leqslant n, t\in[0,\infty), q\in R^n} \sum_{k=1}^n |\sigma_{ik}(t,q)\sigma_{jk}(t,q)|$.

(iii) For any $h > 0$ and $T > 0$ there exists $c_2 = c_2(c_1, n)$ such that

$$P\left\{\max_{0\leqslant t\leqslant T} |q_t^\mu - q_t| > h\right\} \leqslant \frac{\mu}{h^2} T c_2[|p|^2 + \| b \|^2 + \| \sigma\sigma^* \|].$$

Proof. Put $p_t^\mu = \dot{q}_t^\mu$, where q_t^μ is the solution of Eq (7). It follows from (7), that:

$$p_t^\mu = pe^{-\frac{t}{\mu}} + \frac{1}{\mu}e^{-\frac{t}{\mu}}\int_0^t e^{\frac{s}{\mu}} b(s,q_s^\mu)ds + \frac{1}{\mu}e^{-\frac{t}{\mu}}\int_0^t e^{\frac{s}{\mu}}\sigma(s,q_s^\mu)dW_s. \tag{9}$$

We derive from (9), after integrating by part,

$$\begin{aligned} q_t^\mu &= q + \mu p(1 - e^{-\frac{t}{\mu}}) \\ &\quad + \int_0^t b(s,q_s^\mu)ds + \int_0^t \sigma(s,q_t^\mu)dW_s - e^{-\frac{t}{\mu}}\int_0^t b(s,q_s^\mu)e^{\frac{s}{\mu}}\,ds \\ &\quad - e^{-\frac{t}{\mu}}\int_0^t e^{\frac{s}{\mu}}\sigma(s,q_s^\mu)dW_s. \end{aligned} \tag{10}$$

Since $|e^{-\frac{t}{\mu}}\int_0^t e^{\frac{s}{\mu}}b(s,q_s^{\mu})ds| \leqslant |e^{-\frac{t}{\mu}}\mu\int_0^{\frac{t}{\mu}} e^s b(\mu s, q_{\mu s}^{\mu})ds| \leqslant \| b \| \mu$, using Eqs (8), (10) and the Gronwell lemma, we derive the first statement of Lemma 1.

The second bound follows from (8), (10) and Gronwell lemma, if one takes into account the properties of Ito's stochastic integral. The last statement follows from (ii) and the Kolmogorov-Doob inequality. ∎

Remark. Note, that equality (4) follows from Lemma 1. One can also derive from (10) and (8) the main term of the difference $q_t^{\mu} - q_t$ as $\mu \downarrow 0$.

Let, for brevity, $n=1$, and

$$V_t^{\delta} = \frac{1}{\sqrt{\delta}} \int_0^t \xi_{s/\delta}\, ds.$$

Here ξ_s is a mean zero stationary Gaussian process with a fast enough decreasing smooth correlation function $R(|t|)$, such that $\max_{0\leqslant t\leqslant T}|W_t - V_t^{\delta}| \to 0$ as $\delta \downarrow 0$ with probability 1. As is known,[8] for any $\mu > 0$ the solution $(p_t^{\mu,\delta}, q_t^{\mu,\delta})$of the system

$$\mu \dot{p}_t^{\mu,\delta} = b(q_t^{\mu,\delta}) - p_t^{\mu,\delta} + \sigma(q_t^{\mu,\delta})\frac{dV_t^{\delta}}{dt}, \qquad \dot{q}_t^{\mu,\delta} = p_t^{\mu,\delta}, \qquad q_0^{\mu,\delta} = q, \qquad p_0^{\mu,\delta} = p, \tag{11}$$

converges to the solution of (2) as $\delta \downarrow 0$. Thus, a positive function $\tilde{f} = \tilde{f}(\mu, h)$ exists such that

$$P\left\{\max_{0\leqslant t\leqslant T}|q_t^{\mu,\delta} - q_t^{\mu}| > \frac{h}{2}\right\} \leqslant \frac{h}{2} \tag{12}$$

if $\delta \leqslant \tilde{f}(\mu, h)$. One can write down $\tilde{f}(\mu, h)$ in an explicit form, but it is not our goal here.

The function q_t^{μ} in (2) has a continuous derivative p_t^{μ}. Therefore the Ito integral and the Stratouovich integral in (2) coincide, and one can consider (2) as the Ito equation. On the other hand, it follows from the last statement of Lemma 1, that

$$P\left\{\max_{0\leqslant t\leqslant T}|q_t^{\mu} - q_t| > \frac{h}{2}\right\} \leqslant c_3\frac{4\mu}{h^2}, \tag{13}$$

where $c_3 = Tc_2[p^2 + \| b \|^2 + \| \sigma\sigma^* \|]$.

Put

$$f(\mu) = \tilde{f}(\mu, 2(c_3\mu)^{1/3}).$$

Then (12) and (13) imply

$$P\left\{\max_{0\leqslant t\leqslant T} |q_t^{\mu,\delta} - q_t| > h\right\} \leqslant P\left\{\max_{0\leqslant t\leqslant T} |q_t^{\mu,\delta} - q_t^{\mu}| > \frac{h}{2}\right\} + P\left\{\max_{0\leqslant t\leqslant T} |q_t^{\mu} - q_t| > \frac{h}{2}\right\} \leqslant h$$

if $\delta < f(\mu)$ and $\mu < h^3/8c_3$. This means that $q_t^{\mu,\delta}$ converges to the solution of (3) where the stochastic term is understood in the Ito sense, if $\mu, \delta \downarrow 0$ and $\delta < f(\mu)$.

Let us show now that if $\mu, \delta \downarrow 0$ and μ tends to zero fast enough compared to δ, then $q_t^{\mu,\delta}$ converges to the solution of (3) with Stratonouvich's stochastic term.

Put

$$\tilde{b}(t,q) = b(q) + \sigma(q)\frac{dV_t^{\delta}}{dt} = b(q) + \frac{1}{\sqrt{\delta}}\sigma(q)\xi_{t/\delta}.$$

It follows from the bounds for maximum of a stationary process (see ref. 13, Ch. 12), that

$$\lim_{T\to\infty} P\left\{\max_{0\leqslant t\leqslant T} |\xi_t| < c\sqrt{\ln T}\right\} = 1$$

for some constant $c < \infty$ which is defined by the correlation function. Thus, for $0 \leqslant t \leqslant T$

$$|\tilde{b}(t,q)| \leqslant \| b \| + c_4 \| \sigma \| \frac{\sqrt{|\ln \delta|}}{\sqrt{\delta}},$$
$$|\tilde{b}(t,q_1) - \tilde{b}(t,q_2)| \leqslant \tilde{K}|q_1 - q_2|,$$

where

$$\tilde{K} = c_4 K \frac{\sqrt{|\ln \delta|}}{\sqrt{\delta}},$$

with probability close to 1 as δ is small enough. Then, applying the first statement of Lemma 1, we have

$$\max_{0\leqslant t\leqslant T}|q_t^{\mu,\delta}-q_t^{0,\delta}|\leqslant e^{\tilde{K}T}\mu\sqrt{\frac{|\ln\delta|}{\delta}}=\mu(\delta^{-1}|\ln\delta|)^{\frac{1}{2}}\exp\{c_4KT(\delta^{-1}|\ln\delta|)^{1/2}\}.$$

with probability close to 1 if δ is small enough. Here $q_t^{0,\delta}$ is the solution of the equation

$$\dot{q}_t^{0,\delta}=b(q_t^{0,\delta})+\sigma(q_t^{0,\delta})\dot{V}_t^{\delta},\quad q_0^{0,\delta}=q.$$

The last bound implies, that

$$\lim_{\mu,\delta\downarrow 0}\max_{0\leqslant t\leqslant T}|q_t^{\mu,\delta}-q_t^{0,\delta}|=0$$

in probability if μ and δ tend to zero so that lim μ $\exp\{\frac{1}{\delta}\}=0$.

On the other hand, as is shown in Ref. 8, $q_t^{0,\delta}$ converges in probability uniformly on $[0,T]$ to the solution of Stratonovich's equation (5) as $\delta\downarrow 0$.

We sum up our result in

Proposition 1. The process $q_t^{\mu,\delta}$ defined by (11) converges in probability uniformly on $[0,T]$ to the solution of (3) with Ito's stochastic term as $\mu,\delta\downarrow 0$ so that $\delta<f(\mu)$. If $\mu,\delta\downarrow 0$ and $\lim\mu e^{1/\delta}=0$, then $q_t^{\mu,\delta}$ converges to the solution of (3) with Stratonovich's stochastic term.

In particular, if, first, $\delta\downarrow 0$ and then $\mu\downarrow 0$, the limit is the solution of Ito's equation (3). If, first $\mu\downarrow 0$ and then $\delta\downarrow 0$, $q_t^{\mu,\delta}$ converges to the solution of Eq (3) in the Stratonovich sense.

3. STATIONARY DISTRIBUTION

Here we shortly consider relation between stationary distributions for processes q_t^{μ} and q_t defined by Eqs (2) and (3), respectively.

Consider first the case of linear oscillator with one degree of freedom perturbed by the white noise

$$\mu\ddot{q}_t^{\mu}+\dot{q}_t^{\mu}+q_t^{\mu}=\dot{W}_t. \tag{14}$$

One can conclude from (14) that the stationary distribution is the mean zero Gaussian distribution with the variance

$$\sigma_\mu^2=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{d\lambda}{(1-\mu\lambda^2)^2+\lambda^2}. \tag{15}$$

It is clear from this formula, that $\lim_{\mu\downarrow 0}\sigma_\mu^2=(1/\pi)\int_{-\infty}^{+\infty}d\lambda/(1+\lambda^2)=1$. Thus the stationary distribution for q_t^{μ} converges to the stationary distribution of q_t as $\mu \downarrow 0$. But, actually, integral (15) can be calculated explicitly using the residue theory, and it turns out that $\sigma_\mu^2 = 1$ for any $\mu > 0$. So that the stationary distribution for q_t^μ defined by (14) coincides with stationary distribution of q_t for any $\mu > 0$.

This is a manifestation of a more general result: If the field $b(q)$ is potential with respect to white noise, so that the system has the form

$$\dot{p}_t^\mu = -\frac{1}{\mu}\nabla F(q_t^\mu) - \frac{1}{\mu}p_t^\mu + \frac{1}{\mu}\dot{W}_t, \qquad \dot{q}_t^\mu = p_t^\mu, \qquad p_0^\mu = p, \qquad q_0^\mu = q, \tag{16}$$

then the invariant density of q_t has the form $C^{-1}\exp\{-2F(q)\}$, provided that $C = \int_{\mathbb{R}^n}\exp\{-2F(q)\}dq < \infty$. This follows from a well-known fact that the Boltzmann distribution $A_\mu \exp\{-(\mu^2 p^2 + 2F(q))\}$ is invariant for the $2n$-dimensional Markov process (p_t^μ, q_t^μ).

4. HOMOGENIZATION

Consider the process $q_t^{\mu,\varepsilon}$ defined by equations

$$\mu\dot{p}_t^{\mu,\varepsilon} = b(q_t^{\mu,\varepsilon}\varepsilon^{-1}) + \sigma(q_t^{\mu,\varepsilon}\varepsilon^{-1})\dot{W}_t - p_t^{\mu,\varepsilon}, \quad \dot{q}_t^{\mu,\varepsilon} = p_t^{\mu,\varepsilon}, \quad p_0^{\mu,\varepsilon} = p, \quad q_0^{\mu,\varepsilon} = q. \tag{17}$$

If the coefficients $b(q)$ and $\sigma(q)$ are periodic or form a space-homogeneous random field with good enough mixing properties and the matrix $a(q) = \sigma(q)\sigma^*(q)$ is uniformly non-degenerated, one can expect that for $0 < \varepsilon \ll 1$ the process $q_t^{\mu,\varepsilon}$ is close in the weak topology to the solution of Eq (17) with b and σ replaced by constant $\bar{b}$ and $\bar{\sigma}$. Such an approximation for Eq (3) and for corresponding PDEs was studied intensively.[14–16]

In particular, if $b(q)$ and $\sigma(q)$ are 1-periodic in each variable, the effective coefficients $\bar{b}$ and $\bar{\sigma}$ for Eq (3) should be calculated in the following way. Consider the diffusion process on n-dimensional unit torus T^n governed by the equation

$$\dot{\tilde{q}}_t = \sigma(\tilde{q})\dot{W}_t.$$

Since the matrix $a(q) = \sigma(q)\sigma^*(q)$ is assumed to be non-degenerate, such a process on T^n has a unique invariant measure, and its density $m(q)$ is the unique normalized solution of the Fokker–Plank (Forward Kolmogorov) equation

$$\sum_{i,j=1}^{n} \frac{\partial^2}{\partial q^i \partial q^j}(a^{ij}(q)m(q))=0, \qquad \int_{T^n} m(q)dq=1.$$

Then the effective $\bar{b}$ and $\bar{\sigma}$ are given as follows([14]):

$$\bar{b}=\int_{T^n} b(q)m(q)=1, \qquad \bar{\sigma}=\bar{a}^{\frac{1}{2}}, \qquad \bar{a}=\int_{T^n} a(q)m(q)dq.$$

In particular, for $n = 1$, $m(q) = \frac{1}{a(q)}(\int_0^1 \frac{dq}{a(q)})^{-1}$, $\bar{a} = (\int_0^1 \frac{dq}{a(q)})^{-1}$, $\bar{b}=\int_0^1 \frac{b(q)}{a(q)}dq \cdot \bar{a}$.

If K is the Lipschitz constant for $b(q)$ and $\sigma(q)$, then the Lipschitz constant for $b(q\varepsilon^{-1})$ and $\sigma(q\varepsilon^{-1})$ is $K\varepsilon^{-1}$. Let $q_t^{0,\varepsilon}$ be the solution of the equation

$$\dot{q}_t^{0,\varepsilon}=b(q_t^{0,\varepsilon}\varepsilon^{-1})+\sigma(q_t^{0,\varepsilon}\varepsilon^{-1})\dot{W}_t, \quad q_0^{0,\varepsilon}=q. \tag{18}$$

It follows from Lemma 1, that

$$\max_{0\leqslant t\leqslant T} |q_t^{\mu,\varepsilon}-q_t^{0,\varepsilon}| \to 0 \tag{19}$$

in probability as $\mu,\varepsilon \downarrow 0$ so that, for any $c>0$, $\mu\exp\{c\varepsilon^{-2}\}\to 0$. On the other hand, process $q_t^{0,\varepsilon}$ converges as $\varepsilon\downarrow 0$(weakly in the space of continuous functions on $[0,T]$) to the Gaussian process $\bar{q}_t$,

$$\dot{\bar{q}}_t=\bar{b}+\bar{a}^{\frac{1}{2}}W_t, \qquad \bar{q}_0=q$$

with $\bar{b}$ and $\bar{a}$ defined above. This statement together with (19) implies that $q_t^{\mu,\varepsilon}$ converges weakly to $\bar{q}_t$ as $\mu,\varepsilon\downarrow 0$ so that, for any $c>0$, $\mu\exp\{c\varepsilon^{-2}\}\to 0$.

We will show now, that if $\varepsilon,\mu\downarrow 0$ and ε tends to zero much faster than μ, the weak limit of $q_t^{\mu,\varepsilon}$ is different. First we need some auxiliary bounds. We present them, for brevity, in the one-dimensional case.

Lemma 2. Let $(p_t^{\mu,\varepsilon}, q_t^{\mu,\varepsilon})$ be the solution of Eq (17), $n=1$.

(i) For any $T>0$,

$$\lim_{\mu\downarrow 0} P_{p,q}\left\{\max_{0\leqslant t\leqslant T} |p_t^{\mu,\varepsilon}| > \frac{\sqrt{\ln\mu^{-1}}}{\mu}\right\}=0. \tag{20}$$

(ii) There exists $c>0$ such that

$$E_{p,q}|p_t^{\mu,\varepsilon}-p|^4 \leqslant c\mu^{-4}t^2. \tag{21}$$

To prove the first statement, note that the measure $M^{\mu,\varepsilon}$ in the space of trajectories of the process $(p_t^{\mu,\varepsilon}, q_t^{\mu,\varepsilon})$ on the time interval $[0,T]$ is absolutely continuous with respect the measure $\bar{M}^{\mu,\varepsilon}$ corresponding to the process with $b(q)\equiv 0$.

The density

$$\frac{dM^{\mu,\varepsilon}}{d\bar{M}^{\mu,\varepsilon}} = \exp\left\{\int_0^T \sigma^{-1} b\, dW_s - \frac{1}{2}\int_0^T (\sigma^{-1}b)^2 ds\right\}$$

is independent of μ. Using the random change of time and comparison arguments, one can see that it is sufficient to prove (20) for the process $\bar{p}_t$ such that

$$\dot{\bar{p}}_t^{\mu} = -\frac{\tilde{c}}{\mu}\bar{p}_t^{\mu} + \frac{1}{\mu}\dot{W}_t, \quad \bar{p}_0^{\mu} = p,$$

where $\tilde{c}$ is an appropriate positive constant. Bound (20) for process $\bar{p}_t^{\mu}$ can be derived from the iterated logarithm law.

The second statement of Lemma 2 follows from (9) and the standard bounds for stochastic integrals.

Lemma 3. The family of stochastic process $q_t^{\mu,\varepsilon}, 0\leqslant t\leqslant T$, is weakly tight in C_{0T}.

Proof. It follows from (17), that

$$\begin{aligned} q_{t+h}^{\mu,\varepsilon} - q_t^{\mu,\varepsilon} &= \int_t^{t+h} p_s^{\mu,\varepsilon}\, ds = -\mu(p_{t+h}^{\mu,\varepsilon} - p_t^{\mu,\varepsilon}) \\ &\quad + \int_t^{t+h} b(\varepsilon^{-1}q_s^{\mu,\varepsilon})ds + \int_t^{t+h}\sigma(\varepsilon^{-1}q_s^{\mu,\varepsilon})dW_s. \end{aligned}$$

Taking into account (21), this equality implies, that a constant C exists such that

$$E_{p,q}|q_{t+h}^{\mu,\varepsilon} - q_t^{\mu,\varepsilon}|^4 \leqslant Ch^2.$$

The last bound, as is known, provides weak tightness. ∎

Lemma 4. Let $\chi_{[-u,u]}(q)$ be the indicator function of $[-u,u]\subset R^1$, $T>0$. A constant C exists such that, for any $\mu,\varepsilon\in(0,1]$,

$$E\int_0^T \chi_{[-u,u]}(p_s^{\mu,\varepsilon})ds \leqslant Cu^{1/2}\mu^{1/4} \tag{22}$$

Proof. If $b\equiv 0$ and $\sigma\equiv 1$, $p_t^{\mu,\varepsilon}$ is a Markovian Guassian process. One can write down its transition density explicitly and obtain the bound

$$E\int_0^T \chi[-u,u](P_s^{\mu,\varepsilon})ds \leqslant \tilde{C}u\mu^{1/2}.$$

This bound implies (22), if one takes into account that the general case can be reduced to the case $b\equiv 0$ and $\sigma\equiv 1$ by a random time change and absolutely continuous change of measure in the space of trajectories. ∎

Proposition 2. Let $(p_t^{\mu,\varepsilon}, q_t^{\mu,\varepsilon})$ be the solution of system (17). Assume that the functions $b(q)$ and $\sigma(q)$ are 1-periodic in each variable, twice continuously differentiable, and the matrix $a(q)=(a^{ij}(q))=\sigma(q)\sigma^*(q)$ is non-degenerate. Let T^n be n-dimensional unit torus.

(i) Suppose that $\mu,\epsilon\downarrow 0$ so that for any $C>0$, $\mu\exp\{C\varepsilon^{-2}\}\to 0$.

Then, for any $T>0$, process $q_t^{\mu,\varepsilon}$ converge weakly in C_{0T} to the Gaussian Markov process

$$\bar{q}_t = q+\bar{b}t+\bar{\sigma}W_t.$$

Here $\bar{b}=\int_{T^n} b(q)m(q)dq$, $\bar{\sigma}=\bar{(a)}^{1/2}$, $\bar{a}=\int_{T^n} a(q)m(q)dq$, where $m(q)$ is the unique solution of the problem

$$\sum_{i,j=1}^{n} \frac{\partial^2}{\partial q^i\partial q^j}(a^{ij}(q)m(q))=0, \qquad \int_{T^n} m(q)dq=1.$$

(ii) Suppose that $\mu,\varepsilon\downarrow 0$ so that $\varepsilon\mu^{-2}\ln^2\mu\to 0$. Then the process $q_t^{\mu,\varepsilon}$ converge weakly to the Gaussian Markov process,

$$\hat{q}^t = q+\hat{q}t+\hat{\sigma}W_t,$$

where $\hat{b}=\int_{T^n} b(q)dq$, $\hat{\sigma}=(\hat{a})^{1/2}$, $\hat{a}=\int_{T^n} a(q)dq$.

A sketch of the proof. For the first statement was given above. A sketch of the proof for the second statement we give in one dimensional case. Choose $\Delta>0$ so small that, on any interval of length Δ belonging to $[0,T]$, $q_t^{\mu,\varepsilon}$ changes just a little with probability close to 1.

To choose such a Δ, note that the first of Eq. (17) implies:

$$p^{\mu,\varepsilon}_{t_0+t} - p^{\mu,\varepsilon}_{t_0} = -\frac{1}{\mu}\int_{t_0}^{t_0+t} p^{\mu,\varepsilon}_s \, ds + \frac{1}{\mu}\int_{t_0}^{t_0+t} b(q^{\mu,\varepsilon}_s(\varepsilon^{-1}q^{\mu,\varepsilon}_s)\, ds + \frac{1}{\mu}\int_{t_0}^{t_0+t} \sigma(b(q^{\mu,\varepsilon}_s\varepsilon^{-1})dW_s.$$

Using Lemma 2i and Levy's Hölder continuity of the Wiener process (see ref. 17, §1.9), we conclude from the last equality, that

$$\max_{t_0\in[0,T],0<t<\Delta} |P^{\mu,\varepsilon}_{t_0+t} - P^{\mu,\varepsilon}_{t_0}| < C[\frac{\Delta\sqrt{\ln\mu^{-1}}}{\mu^2} + \frac{\sqrt{\Delta\ln\Delta^{-1}}}{\mu}]$$

with probability close to 1 as Δ and μ are small enough. This inequality implies that

$$\max_{t_0\in[0,T],0<t<\Delta} |P^{\mu,\varepsilon}_{t_0+t} - P^{\mu,\varepsilon}_{t_0}| \ll 1, \quad if\ \Delta = \frac{\mu^2}{\ln^2\mu} \text{ and } \mu << 1.$$

This means, that on any time interval of length Δ, $\dot{q}^{\mu,\varepsilon}_t = p^{\mu,\varepsilon}_t$ is close to a constant. Therefore, taking into account (22), for any continuous 1-periodic function $f(q)$

$$\frac{1}{\Delta}\int_{t_0}^{t_0+\Delta} f(q^{\mu,\varepsilon}_s\varepsilon^{-1})ds \longrightarrow \int_0^1 f(q)dq \tag{23}$$

as $\mu \downarrow 0$, and $\varepsilon \ll \Delta = \mu^2/\ln^2\mu << 1$. Convergence of finite-dimensional distributions of the process $q^{\mu,\varepsilon}_s$ to $\hat{q}_t$ can be derived from (23). This together with Lemma 3 implies weak convergence of $q^{\mu,\varepsilon}_s$ to $\hat{q}_t$ in C_{0T}.

Remark. In the multidimensional case, the proof of (23) is more sophisticated because the dynamical system $\dot{q} = p$ on T^n, where p is a constant vector, can have non-unique invariant measure. To overcome this difficulty, one should use the fact that, inspite of the degeneration, the process $(p^{\mu,\varepsilon}_t, q^{\mu,\varepsilon}_t)$ has a positive transition density and spends time zero in the set where the first coordinate has rationally dependent components (compare with ref. 18).

5. LARGE DEVIATIONS

Relations between large deviation problems for systems (2) and (3) when the noise term is small are considered in this section. In particular, we discuss the exit problem and stochastic resonance.

Let the process $q_t^{\mu,\varepsilon}$ is defined by the system

$$\mu \dot{p}_t^{\mu,\varepsilon} = b(q_t^{\mu,\varepsilon}) - p_t^{\mu,\varepsilon} + \sqrt{\varepsilon}\sigma(q_t^{\mu,\varepsilon})\dot{W}_t, \qquad p_0^{\mu,\varepsilon} = p \in R^n, \tag{24}$$
$$\dot{q}_t^{\mu,\varepsilon} = p_t^{\mu,\varepsilon}, \quad q_0^{\mu,\varepsilon} = q \in R^n, \qquad 0 < \varepsilon \ll 1.$$

For every $\varepsilon > 0$, the Smoluchowski–Kramers approximation $q_t^{(\varepsilon)}$ is given by the equation

$$\dot{q}_t^{(\varepsilon)} = b(q_t^{(\varepsilon)}) + \sqrt{\varepsilon}\sigma(q_t^{(\varepsilon)})\dot{W}_t, \; q_0^{(\varepsilon)} = q. \tag{25}$$

We assume that the coefficients $b(q)$ and $\sigma(q)$ are smooth enough and bounded; $\det a(q) \geqslant a_o > 0$ where $a(q) = \sigma(q)\sigma^*(q)$. As is known,[7] the action functional for the family $q_t^{(\varepsilon)}$ in C_{0T} as $\varepsilon \downarrow 0$ has the form $\varepsilon^{-1} S_{0T}(\varphi)$, where

$$S_{oT}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T (a^{-1}(\varphi_s)(\dot{\varphi}_s - b(\varphi_s)), \dot{\varphi}_s - b(\varphi_s))ds, & \varphi_0 = q, \varphi \text{ is abs. cont.} \\ +\infty, & \text{for the rest of } C_{0T}. \end{cases}$$

Let $K \in R^n$ be an asymptotically stable equilibrium for the dynamical system $q_t^{(0)}$ in R^n defined by (25) with $\varepsilon = 0$, and $G \subset R^n$ be a bounded domain with smooth boundary ∂G attracted to K. Let $b(q) \centerdot n(q)|_{\partial G} < 0$, where $n(q)$ is the exterior normal to ∂G. Denote by $\tau^{(\varepsilon)}$ the first exit time of $q_t^{(\varepsilon)}$ from G: $\tau^{(\varepsilon)} = \min\{t : q_t^{(\varepsilon)} \in \partial G\}$.

Introduce the quasi-potential for process $q_t^{(\varepsilon)}$(with respect to the equilibrium $K \in G \subset R$):

$$V(q) = \inf\{S_{0T}(\varphi) : \varphi \in C_{0T}, \varphi_0 = K, \; \varphi_T = q, \; T > 0\}$$

Then (see ref. 11), $\varepsilon \ln \tau^{(\varepsilon)} \to V_o = \min_{q\in\partial G} V(q)$ in probability, starting from any $q \in G$. If $q^* \in \partial G$ is the unique minimum of $V(q)$ on ∂G, then $q_{\tau^{(\varepsilon)}}^{(\varepsilon)} \to q^*$ in probability as $\varepsilon \downarrow 0$. Moreover, if φ_t^* is the unique (up to a time shift) extremal of the action functional on the set of curves connecting K and q^*, then the last part of the trajectory $q_t^{(\varepsilon)}$ before exiting G belongs to a small neighborhood of φ^* with probability close to 1 as ε is small enough. In the case of unit matrix $a(x)$ and potential field $b(q) = -\nabla F(q)$, $V(q) = 2F(q)$ for $q \in \{x \in G : V(x) \leqslant V_o\}$, if the potential $F(q)$ is normalized by the conditions $F(K) = 0$ (see ref. 11, Ch. 4).

Suppose the non-perturbed system $q_t^{(0)}$ is generic and has several asymptotically stable equilibriums $K_1, \dots, K_m$, so that any point of R^n, besides the separatrices is attracted to one of K_i, then the long-time evolution of $q_t^{(\varepsilon)}$, for $0 < \varepsilon \ll 1$ is defined by the numbers

$$V_{ij} = \inf\{S_{0T}(\varphi) : \varphi_0 = K_i, \varphi_T = K_j,\ T > 0\}.$$

In particular, this numbers define the hierarchy of cycles, metastable states, asymptotics of the transition times, sub-limiting distributions[19]. If b and σ are slowly changing in time, then the numbers V_{ij} also depend on this slow time, and their evolution (or, more precisely, bifurcations in the structure of the hierarchy of cycles) defines stochastic-resonance-type effects.[5]

Consider now the process $q_t^{\mu,\varepsilon}$, $0 \leqslant t \leqslant T$, defined by (24), as μ is fixed and $\varepsilon \downarrow 0$. Taking into account that $q_t^{\mu,\varepsilon}$ is continuously differentiable, one can check that the transformation $W_t \to q_t^{\mu,\varepsilon}$ defined by (24) is continuous in C_{0T}. This, according to results from [11], Section3.3, implies that the action functional for $q_t^{\mu,\varepsilon}$ in C_{0T} as $\varepsilon \downarrow 0$ has the form $\varepsilon^{-1} S_{0T}^{\mu}(\varphi)$, where

$$S_{oT}^{\mu}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T (a^{-1}(\varphi_s)(\mu\ddot{\varphi}_s + \dot{\varphi}_s - b(\varphi_s)), \mu\ddot{\varphi}_s + \dot{\varphi}_s - b(\varphi_s))ds, \\ \text{if } \dot{\varphi} \text{ is abs. cont. and } \varphi_0 = q, \dot{\varphi}_0 = p, \\ +\infty, \quad \text{for the rest of } C_{0T}. \end{cases}$$

We consider in more detail the case of unit diffusion matrix and potential fields $b(x)$: $a(q) = I$, $b(q) = -\nabla F(q)$, $q \in R^n$. Note, that if K is an equilibrium of the system $q_t^{(0)}$, then $(0, K)$ is an equilibrium for system (24) with $\varepsilon = 0$. Moreover, if K is asymptotically stable for $q_t^{(0)}$, $(0, K)$ is asymptotically stable for $(p_t^{\mu,0}, q_t^{\mu,0})$. If a domain $G \subset R^n$ is attracted to K for $q_t^{(0)}$, then for $(p_t^{\mu,0}, q_t^{\mu,0})$, the domain $\{(p, q), q \in G, |p| < A\} \subset R^{2n}$ is attracted to $(0, K) \in R^{2n}$ for any $A > 0$ if $\mu = \mu(A)$ is small enough. One can introduce quasi-potential (with respect to the equilibrium K):

$$V^{\mu}(q) = \inf\left\{S_{0T}^{\mu}(\varphi) : \varphi_0 = k,\ \dot{\varphi}_0 = 0,\ \varphi_T = q,\ T > 0\right\}. \tag{26}$$

The diffusion process $(p_t^{\mu,\varepsilon}, q_t^{\mu,\varepsilon})$ for any $\mu, \varepsilon > 0$, although it is degenerate, has strictly positive for $t > 0$ transition density $P(t; p_o, q_o; p_1, q_1)$. The last remark allows to obtain results concerning the exit problem for $q_t^{\mu,\varepsilon}$ in the way similar to the case of process $q_t^{(\varepsilon)}$.[11] Consider a domain $G \subset R^n$ such that $G \cup \partial G$ is attracted to the equlibrium K for the vector field $b(q)$ and $b(q) \cdot n(q)|_{\partial G} < 0$. Put $\hat{G} = \left\{(p, q) \in R^{2n} : q \in G\right\}$, $\tau^{\mu,\varepsilon} = \min\left\{t : q_t^{\mu,\varepsilon} \in \partial G\right\}$.

Proposition 3. Let $a(q) = \sigma(q)\sigma^*(q)$ be the unit matrix and $b(q) = -\nabla F(q)$, $F(K) = 0$. Let $q \in G$ and the trajectory $(p_t^{\mu,0}, q_t^{\mu,0})$ defined by (24) with $\varepsilon = 0$, $p_0^{\mu,0} = p, q_0^{\mu,\varepsilon} = q$, tends to $(0, K)$ as $t \to \infty$ not leaving the domain G.

Then

$$\lim_{\varepsilon\downarrow 0} \varepsilon \ln \tau^{\mu,\varepsilon} = \lim_{\varepsilon\downarrow 0} \varepsilon \ln E_{p,q}\tau^{\mu,\varepsilon} = \min_{q\in\partial G} V^{\mu}(q). \tag{27}$$

If $V^{\mu}(q)$ has on ∂G a unique minimum at $q^* \in \partial G$, then $q^{\mu,\varepsilon}_{\tau^{\mu,\varepsilon}} \to q^*$ in probability as $\varepsilon \downarrow 0$. For any $\mu > 0$, $q \in \{q \in G : V^{\mu}(q) \leqslant \min_{z\in\partial G} V^{\mu}(z)$,

$$V^{\mu}(q) = 2F(q), \ \min_{z\in\partial G} V^{\mu}(z) = 2 \min_{z\in\partial G} F(z) = V_o, \tag{28}$$

q^* is independent of μ. Thus the logarithmic asymptotic of the exit time and of the exit position as $\varepsilon \downarrow 0$ for the processes $q_t^{\mu,\varepsilon}$ and for its Smoluchowski–Krammers approximation $q_t^{(\varepsilon)}$ are the same.

Proof. of equality (27) is similar to the proof of corresponding statement for $q_t^{(\varepsilon)}$ (ref. 11, Ch. 4), and we omit it. To prove (28), note that

$$\begin{aligned}
&\int_0^T |\mu\ddot{\varphi}_t + \dot{\varphi}_t + \nabla F(\varphi_t)|^2 dt \\
&= \int_0^T |\mu\ddot{\varphi}_t - \dot{\varphi}_t + \nabla F(\varphi_t)|^2 dt + 4\int_0^T \dot{\varphi}_t \cdot \nabla F(\varphi_t) dt + 4\mu \int_0^T \ddot{\varphi}_t \cdot \dot{\varphi}_t dt \\
&= \int_0^T |\mu\ddot{\varphi}_t - \dot{\varphi}_t + \nabla F(\varphi_t)|^2 dt + 4(F(\varphi_T) - F(\varphi_0)) + 2\mu(|\dot{\varphi}_T|^2 - |\dot{\varphi}_0|^2).
\end{aligned} \tag{29}$$

If ψ_t^{μ} is the solution of the problem

$$\mu\ddot{\psi}_t^{\mu} + \dot{\psi}_t^{\mu} + \nabla F(\psi_t^{\mu}) = 0, \qquad \psi_0^{\mu} = q, \qquad \dot{\psi}_0^{\mu} = 0,$$

then the function $\varphi_t^{\mu} = \psi_{T-t}^{\mu}$ satisfies the equation

$$\mu\ddot{\varphi}_t^{\mu} - \dot{\varphi}_t^{\mu} + \nabla F(\varphi_t^{\mu}) = 0, \quad \varphi_T^{\mu} = q, \ \dot{\varphi}_T^{\mu} = 0.$$

It follows from (29), that

$$\int_0^T |\mu\ddot{\varphi}_t^{\mu} + \dot{\varphi}_t^{\mu} + \nabla F(\varphi_t^{\mu})|^2 dt == 4(F(q) - F(\psi_T^{\mu})) + 2\mu(|\dot{\psi}_T^{\mu}|^2). \tag{30}$$

Taking into account that the point $(0, q)$ is attracted to the equilibrium $(0, K)$ of the system $(p_t^{\mu,0}, q_t^{\mu,0})$, we conclude that $\psi_T^{\mu} \to K$ and $|\dot{\psi}_T^{\mu}|^t \to o$ as $T \to \infty$. This remarks together with (26), (29) and (30) implies that $V^{\mu}(q) = 2F(q)$, if we take into account the form of the action functional $S_{0T}^{\mu}(\varphi)$ given above.

Remark. Although the asymptotic exit point and the logatithnic asymptotics of the exit time for $q_t^{(\varepsilon)}$ and $q_t^{\mu,\varepsilon}$ are the same, the extremals are different. So that the last part of trajectory $q_t^{\mu,\varepsilon}$, before it exits domain G, for fixed μ and $\varepsilon << 1$, is situated, after an appropriate time shift (compare with citeFW1, Ch.4), near the extremal defined by the equation

$$\mu\ddot{\hat{\varphi}}_t^\mu - \dot{\hat{\varphi}}_t^\mu + \nabla F(\hat{\varphi}_t^\mu) = 0, \ -\infty < t < 0, \ \hat{\varphi}_0^\mu = q^*, \qquad \dot{\hat{\varphi}}_0^\mu = 0.$$

In particular, in one-dimensional case, the extremal for q_t^ε approaches monotonically one of the ends of the interval G. The extremal for $q_t^{\mu,\varepsilon}$, $\mu > 0$, will oscillate near the equilibrium with increasing amplitude until it hits one of the ends of interval G. The extremals come close as $\mu \downarrow 0$.

Consider now the case of several stable equilibriums of the vector field $b(q)$, $q \in R^n$. Let, as in Proposition 4, $a(q) \equiv I$, $b(q) = -\nabla F(q)$. We assume that $\lim_{|q|\to\infty} F(q) = \infty$, so that each trajectory q_t of the field $b(q)$, excluding separatrices, approaches as $t \to \infty$ one of the local minima $q_1, \dots, q_l \in R^n$ of the potential. System (24) with $\varepsilon = 0$, in this case, has asymptotically stable equiliberiums $(0, q_1), \dots, (0, q_l)$. Let μ be fixed and $\varepsilon \downarrow 0$. We write $T_\varepsilon^\lambda \asymp e^{\frac{\lambda}{\varepsilon}}$, if $\lim_{\varepsilon\downarrow 0} \varepsilon \ln T_\varepsilon^\lambda = \lambda$. For any initial point $x = (p, q) \in R^{2n}$ and any time scale $T_\varepsilon^\lambda \asymp e^{\frac{\lambda}{\varepsilon}}, \lambda > 0$, (excluding a finite set of values λ and initial points belonging to separatrices), one can point out a stable equilibrium $M(x, \lambda) = (0, q_{m(x,\lambda)})$ which is called metastable states for given $x = (p, q)$ and $\lambda > 0$, such that the trajectory $q_t^{\mu,\varepsilon}$ spends most of the time during time interval $[0, T_\varepsilon^\lambda]$ in a small neighborhood of $q_{m(x,\lambda)}$. More precisely: in generic case, there exists one equilibrium $q_{m(x,\lambda)}$ such that for any $\delta > 0$,

$$\lim_{\varepsilon\downarrow 0} \frac{1}{T_\varepsilon^\lambda} \int_0^{T_\varepsilon^\lambda} \chi_\delta(q_t^{\mu,\varepsilon}) dt = 1,$$

where χ_δ is the indicator function of δ-neighborhood of $q_{m(x,\lambda)}$ in R^n.[11,19]

The state $M(x, \lambda)$ is defined by the numbers

$$V_{ij}^\mu = \inf \left\{ S_{0T}^\mu(\varphi) : \varphi_0 = q_i, \quad \dot{\varphi}_0 = 0, \quad \varphi_T = q_j, \quad T > 0 \right\}$$

and by the initial point $x = (p, q)$.

Suppose that the trajectory $(p_t^{\mu,0}, q_t^{\mu,0})$ starting at an initial point (p, q) is attracted to the equilibrium $(0, q_k)$ such that the trajectory q_t of the system $\dot{q}_t = b(q_t), q_0 = q$, is attracted to the equilibrium q_k of the field

$b(q)$ in R^n. Note, that for any p there exists $\mu_o > 0$ such that this assumption is satisfied for $\mu < \mu_o$. Moreover, this is true for any $\mu > 0$, if $|p| < \rho_o$ for some $\rho_o = \rho_o(q)$. One can derive from Proposition 4, that the numbers V_{ij}^{μ} are actually independent of μ. Therefore, if $(p_t^{\mu,0}, q_t^{\mu,0})$, $p_0^{\mu,0} = p, q_0^{\mu,0} = q$ is attracted to $(0, q_k)$ and q_t is attracted to q_k, then metastable states for $q_t^{\mu,\varepsilon}, q_0^{\mu,\varepsilon} = q, \dot{q}_0^{\mu,\varepsilon} = p$, for any time scale T_ε^λ, coincide with those for Smoluchowski–Krammers approximation $q_t^{(\varepsilon)}, q_0^{(\varepsilon)} = q$.

Finally, I will make a short remark concerning stochastic resonance for the original system and its Smoluchowski–Krammers approximation.

Suppose that the field b is slowly changing: $b = -\nabla_q F(\frac{t}{T_\varepsilon^\lambda}, q)$. Let, for brevity, the potential $F(z,q)$ be 1-periodic in z, $F(z,q) = F_1(q)$ for $0 \leqslant z < t_1 < 1$ and $F(z,q) = F_2(q)$ for $t_1 \leqslant z < 1$. Moreover, let $F_1(q)$ and $F_2(q)$ have the same minimum points $q_1, \dots, q_l$. Denote by $M_i(q,\lambda)$ the metasatble state for the process $\tilde{q}_t^{(\varepsilon)}$,

$$\dot{\tilde{q}}_t^{(\varepsilon)} = -\nabla F_i(\tilde{q}_t^{(\varepsilon)}) + \sqrt{\varepsilon}\dot{W}_t, \qquad \tilde{q}_0^{(\varepsilon)} = q \in R^n$$

in the time interval $[0, \exp\{\frac{\lambda}{\varepsilon}\}]$. Let the initial conditions $q_0^{\mu,\varepsilon}, \dot{q}_0^{\mu,\varepsilon}$ be such that $(p_t^{\mu,\varepsilon}, q_t^{\mu,\varepsilon})$ is attracted to $(0, q_k), k \in \{1, \dots, l\}$. It follows from our considerations , that $q_t^{\mu,\varepsilon}$ during time interval $[0, t_1 T_\varepsilon^\lambda]$, $T_\varepsilon^\lambda \asymp \exp\{\frac{\lambda}{\varepsilon}\}$, spends most of the time near $M^{(0)} = M_1(q_k, \lambda)$. Then the trajectory $q_t^{\mu,\varepsilon}$ switches to $M^{(1)} = M_2(M^{(0)}, \lambda) = M_2(M_1(q_k,\lambda),\lambda)$ for $t \in [t_1 T_\varepsilon^\lambda, T_\varepsilon^\lambda]$. Then, for $t \in [T_\varepsilon^\lambda, (1+t_1)T_\varepsilon^\lambda]$, $q_t^{\mu,\varepsilon}$ spends most of the time near $M^{(2)} = M_1(M^{(1)}, \lambda)$ and, for $t \in [(1+t_1)T_\varepsilon^\lambda, 2T_\varepsilon^\lambda]$, near $M^{(3)} = M_2(M^{(2)}, \lambda)$ and so on. Each $M^{(k)}$ is one of the equilibriums $q_1, \dots, q_l$. Since the number of the equilibriums is finite sequence $M^{(k)}$ starts to repeat itself. This means that the trajectory $q_t^{\mu,\varepsilon}$ for ε small enough is close (say, in the L^2-norm) to a periodic step function which has values in the set $\{q_1, \dots, q_l\}$. The period of these oscillations is an integer which can be greater than 1 but always less than the number of stable equilibriums l. Since the metastable states for $q_t^{\mu,\varepsilon}$ and $q_t^{(\varepsilon)}$ are the same, the trajectories $q_t^{\mu,\varepsilon}$ and $q_t^{(\varepsilon)}$ are close to the same periodic function. Thus the stochastic-resonance-type effects for $q_t^{\mu,\varepsilon}$ and $q_t^{(\varepsilon)}$ are the same.

If the field $b(q)$is not potential, the quasi-dterministic approximation [19] for system (24) with stable attractors $(0, q_1), \dots, (0, q_l)$ is defined by the numbers V_{ij}^μ introduced above. In general, $V_{ij}^\mu \neq V_{ij}$ for $\mu > 0$, but one can prove that $\lim_{\mu \downarrow 0} V_{ij}^\mu = V_{ij}$. This implies that (in generic case) there exists $\mu_0 > 0$ such that, for $0 < \mu < \mu_0$, the quasi-deterministic approximation for $q_t^{\mu,\varepsilon}$ and $q_t^{(\varepsilon)}$ is the same. Therefore, if $b(q)$ is slowly changing in time, stochastic-resonance-type effects for $q_t^{\mu,\varepsilon}$ and $q_t^{(\varepsilon)}$ will be, in a sense, also the same.

A similar approach allows to describe relations between large deviation effects, such as stochastic resonance, for $q_t^{\mu,\varepsilon}$ and $q_t^{(\varepsilon)}$, if the white noise in (24) is replaced by some other types of stochastic processes (compare with ref. 19).

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